

Spinor wave equation of photon

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In this paper, we give the spinor wave equations of free and unfree photon, which are the differential equation of space-time one order. For the free photon, the spinor wave equations are covariant, and the spinors ψ are corresponding to the the reducibility representations $D^{10} + D^{01}$ and $D^{10} + D^{01} + D^{\frac{1}{2}\frac{1}{2}}$ of the proper Lorentz group.

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1. Introduction

Photons are quantum particles that their behavior is governed by the laws of quantum mechanics. This means, their state are described by wave functions. According to modern quantum field theory, photons, together with all other particles, are the quantum excitations of a field. In the case of photons, these are the excitations of the electromagnetic field. The lowest field excitation of a given type corresponds to one photon and higher field excitations involve more than one photon. This concept of a photon enables one to use the photon wave function not only to describe quantum states of an excitation of the free field but also of the electromagnetic field interacting with a medium. Maxwell equations in the matrix Dirac-like form considered during long time by many authors, the interest to the Majorana-Oppenheimer formulation of electrodynamics has grown in recent years [1-14].

After discovering the relativistic equation for a particle with spin 1/2 [15], In Refs. [1-3], the author have proposed to consider the Maxwell theory of electromagnetism as the wave mechanics of the photon, then it must be possible to write Maxwell equations as a Dirac-like equation for a probability quantum wave $\bar{\psi}$, this wave function being expressable by means of the physical E , B fields, and the complex 3-vector wave function satisfying the massless Dirac-like equations. Afterwards, much work was done to study spinor and vectors within the Lorentz group theory: Moglich [16], Ivanenko-Landau [17], Neumann [18], van der Waerden [19]. As was shown any quantity which transforms linearly under Lorentz transformations is a spinor. For that reason spinor quantities are considered as fundamental in quantum field theory and basic equations for such quantities should be written in a spinor form. A spinor formulation of Maxwell equations was studied by Laporte and Uhlenbeck [20-22]. In this paper, we give the spinor wave equations of free and unfree photon, which are the differential equation of space-time one order. For the free photon, the spinor wave equations are covariant, and the spinors ψ are corresponding to the the reducibility representations of the proper Lorentz group.

2. Spinor wave equation of free photon

The differential equation of space-time two order for free electromagnetism wave is

$$\square A_\mu = 0, \quad (1)$$

and the Lorentz condition is

$$\partial_\mu A_\mu = 0, \quad (2)$$

where

$$\partial_\mu = (\nabla, \frac{\partial}{\partial(ict)}), \quad \square = \partial_\mu \partial_\mu = \nabla^2 - \frac{1}{c^2} \frac{1}{\partial t^2}. \quad (3)$$

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The equation (1) has four components $A_\mu (\mu = 1, 2, 3, 4)$. making Eq. (1) into one order differential equation of space-time, the number of field functions should be added.
Defining the function

$$F_{\nu\mu} = \partial_\nu A_\mu, \quad (4)$$

and Eq. (1) becomes

$$\partial_\nu F_{\nu\mu} = \partial_\nu \partial_\nu A_\mu = \square A_\mu = 0, \quad (5)$$

i.e., the differential equation of space-time one order for free electromagnetism wave is

$$\partial_\nu F_{\nu\mu} = 0. \quad (6)$$

By Eqs. (2) and (4), we have

$$\partial_\mu F_{\nu\mu} = \partial_\nu \partial_\mu A_\mu = 0, \quad (7)$$

with Eqs. (6) and (7), there is

$$\partial_\nu F_{\nu\mu} = \partial_\mu F_{\nu\mu} = \partial_\nu F_{\mu\nu} = 0. \quad (8)$$

From Eq. (8), we have

$$F_{\mu\nu} = F_{\nu\mu}, \quad (9)$$

or

$$F_{\mu\nu} = -F_{\nu\mu}. \quad (10)$$

In Eq. (9), $F_{\mu\nu}$ is symmetry tensor, it has ten independent components, and its matrix is

$$F_{\mu\nu} = \begin{pmatrix} F_{11} & F_{12} & F_{13} & F_{14} \\ F_{12} & F_{22} & F_{23} & F_{24} \\ F_{13} & F_{23} & F_{33} & F_{34} \\ F_{14} & F_{24} & F_{34} & F_{44} \end{pmatrix}, \quad (11)$$

In Eq. (10), $F_{\mu\nu}$ is antisymmetry tensor, it has six independent components, and its matrix is

$$F_{\mu\nu} = \begin{pmatrix} 0 & F_{12} & F_{13} & F_{14} \\ -F_{12} & 0 & F_{23} & F_{24} \\ -F_{13} & -F_{23} & 0 & F_{34} \\ -F_{14} & -F_{24} & -F_{34} & 0 \end{pmatrix}. \quad (12)$$

For the proper Lorentz group L_p , the irreducibility representations of spin 1 particle or field are D^{10} , D^{01} and $D^{\frac{1}{2}\frac{1}{2}}$, respectively, and the dimension of irreducibility representations are corresponding to three, three and four. Therefore, the reducibility representations of particle or field with spin 1 are

$$D = D^{10} + D^{01}, \quad (13)$$

$$D = D^{10} + D^{01} + D^{\frac{1}{2}\frac{1}{2}}, \quad (14)$$

...

Eqs. (13) and (14) are corresponding to six and ten dimensions irreducibility representations, which are the two lowest dimensional irreducibility representations

When $F_{\mu\nu}$ take the antisymmetry tensor of Eq. (12), which is the representation vector of reducibility representations $D = D^{10} + D^{01}$. Eq. (6) can be written as

$$\mu = 1 : \quad \partial_1 F_{11} + \partial_2 F_{21} + \partial_3 F_{31} + \partial_4 F_{41} = 0, \quad (15)$$

$$\mu = 2 : \quad \partial_1 F_{12} + \partial_2 F_{22} + \partial_3 F_{32} + \partial_4 F_{42} = 0, \quad (16)$$

$$\mu = 3 : \quad \partial_1 F_{13} + \partial_2 F_{23} + \partial_3 F_{33} + \partial_4 F_{43} = 0, \quad (17)$$

$$\mu = 4 : \quad \partial_1 F_{14} + \partial_2 F_{24} + \partial_3 F_{34} + \partial_4 F_{44} = 0, \quad (18)$$

substituting Eq. (12) into (15)-(18), there is

$$\begin{cases} \partial_2 F_{12} + \partial_3 F_{13} + \partial_4 F_{14} = 0 \\ \partial_1 F_{12} - \partial_3 F_{23} - \partial_4 F_{24} = 0 \\ \partial_1 F_{13} + \partial_2 F_{23} - \partial_4 F_{34} = 0 \\ \partial_1 F_{14} + \partial_2 F_{24} + \partial_3 F_{34} = 0 \end{cases} \quad (19)$$

Eq. (19) can be written as the differential form of space-time one order

$$\beta_\mu \partial_\mu \psi = 0, \quad (\mu = 1, 2, 3, 4), \quad (20)$$

where the spinor wave function ψ is

$$\psi = \begin{pmatrix} F_{12} \\ F_{13} \\ F_{14} \\ F_{23} \\ F_{24} \\ F_{34} \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \psi_5 \\ \psi_6 \end{pmatrix}, \quad (21)$$

and the β matrixes are

$$\begin{aligned} \beta_1 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \beta_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\ \beta_3 &= \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \beta_4 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned} \quad (22)$$

When $F_{\mu\nu}$ take the symmetry tensor of Eq. (11), which is the representation vector of reducibility representations $D = D^{10} + D^{01} + D^{\frac{1}{2}\frac{1}{2}}$. Eq. (6) can be written as

$$\begin{cases} \partial_1 F_{11} + \partial_2 F_{12} + \partial_3 F_{13} + \partial_4 F_{14} = 0 \\ \partial_1 F_{12} + \partial_2 F_{22} + \partial_3 F_{23} + \partial_4 F_{24} = 0 \\ \partial_1 F_{13} + \partial_2 F_{23} + \partial_3 F_{33} + \partial_4 F_{34} = 0 \\ \partial_1 F_{14} + \partial_2 F_{24} + \partial_3 F_{34} + \partial_4 F_{44} = 0 \end{cases} \quad (23)$$

Eq. (23) can be written as the differential form of space-time one order

$$\beta_\mu \partial_\mu \psi = 0, \quad (24)$$

where the spinor wave function ψ is

$$\psi = \begin{pmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{14} \\ F_{22} \\ F_{23} \\ F_{24} \\ F_{33} \\ F_{34} \\ F_{44} \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \psi_5 \\ \psi_6 \\ \psi_7 \\ \psi_8 \\ \psi_9 \\ \psi_{10} \end{pmatrix}, \quad (25)$$

and the β matrixes are

[illegible]

When $F_{\mu\nu}$ are taken as antisymmetry and symmetry tensors, we obtain the two kinds of spinor wave equations (20) and (24), which maybe describe the left and right circularly polarized photons.

3. Spinor wave equation of photon in medium

In section 2, we give the spinor wave equation of free photon, but it is more important to study the characteristic of photon in medium, and it is essential to give the spinor wave equation of photon in medium.

In Eq. (20) or (24), the spinor wave equation of free photon can be written as

$$(\beta_4 i\hbar \frac{\partial}{\partial t} + ic \cdot i\hbar \vec{\beta} \cdot \vec{\nabla})\psi = 0, \quad (27)$$

For a free photon, the relation between E and momentum p is

$$E = cp, \quad (28)$$

the spinor wave equation (27) is obtained by quantizing equation (28).

For a unfree photon, the relation between E and momentum p is

$$E - V = cp \quad (29)$$

where V is the interaction potential between photon and medium.

Comparing equation (28) with (29), and with the aid of equation (27), we can obtain the quantized equation of (29), it is

$$(\beta_4(i\hbar \frac{\partial}{\partial t} - V) + ic \cdot i\hbar \vec{\beta} \cdot \vec{\nabla})\psi = 0. \quad (30)$$

Eq. (30) is the spinor wave equation of photon in medium.

The interaction potential between photon and medium is [3]

$$V = \frac{hc}{\lambda}(\frac{1}{n} - 1) = h\nu(1 - n), \quad (31)$$

where ν is photon frequency, and n is medium refractive indexes.

Substituting Eq. (31) into (30), there is

$$(\beta_\mu \partial_\mu + \frac{2\pi}{\lambda} \beta_4(1 - n))\psi = 0. \quad (32)$$

When refractive indexes $n = 1$, Eq. (32) becomes free photon equation (22) or (24).

4. The covarience of spinor wave equation

The relativistic quantum theory and quantum field theory should be covariant, the spinor wave equations (20) and (24) should be also.

At Lorentz transformation

$$x'_\mu = \alpha_{\mu\nu} x_\nu, \quad (33)$$

to get

$$x_\mu = \alpha_{\nu\mu} x'_\nu \quad (34)$$

and

$$\partial_\mu = \alpha_{\nu\mu} \partial'_\nu, \quad (35)$$

substituting Eq. (35) into (20) or (24), there is

$$\alpha_{\nu\mu} \beta_\mu \cdot \partial'_\nu \psi = 0, \quad (36)$$

defining

$$\alpha_{\nu\mu}\beta_\mu = L^{-1}\beta_\nu L, \quad (37)$$

i.e.,

$$L\alpha_{\nu\mu}\beta_\mu L^{-1} = \beta_\nu. \quad (38)$$

Eq. (36) becomes

$$L^{-1}\beta_\nu\partial'_\nu L\psi = 0, \quad (39)$$

defining

$$\psi'(x') = L\psi. \quad (40)$$

Eq. (39) becomes

$$\beta_\nu\partial'_\nu\psi'(x') = 0. \quad (41)$$

The covariance of Eqs. (20) and (24) are proved, and we can give the transformation L .

For a infinitesimal Lorentz transformation

$$x_\mu \rightarrow x'_\mu = (\delta_{\mu\nu} + \varepsilon_{\mu\nu})x_\nu, \quad (42)$$

where

$$\varepsilon_{\mu\nu} = -\varepsilon_{\nu\mu}, \quad |\varepsilon_{\mu\nu}| \ll 1, \quad (43)$$

writing Eq. (42) with infinitesimal operator, it is

$$x'_\mu = (\delta_{\mu\nu} + \frac{1}{2}\varepsilon_{\rho\sigma}I_{\rho\sigma}^{\mu\nu})x_\nu, \quad (44)$$

comparing equation (42) with (44), there is

$$\frac{1}{2}\varepsilon_{\rho\sigma}I_{\rho\sigma}^{\mu\nu} = \varepsilon_{\mu\nu}, \quad (45)$$

under the infinitesimal Lorentz transformation, the spinor infinitesimal transformation is

$$\psi'(x') = (1 + \frac{1}{2}\varepsilon_{\rho\sigma}I_{\rho\sigma})\psi(x) \quad (46)$$

where $I_{\rho\sigma}$ is the infinitesimal operator (matrix) of Lorentz group.

Comparing equation (40) with (46), we obtain the transformation matrix L

$$L = 1 + \frac{1}{2}\varepsilon_{\rho\sigma}I_{\rho\sigma} \quad (47)$$

and

$$L^{-1} = 1 - \frac{1}{2}\varepsilon_{\rho\sigma}I_{\rho\sigma}, \quad (48)$$

substituting Eqs. (47) and (48) into (38), there is

$$(1 + \frac{1}{2}\varepsilon_{\rho\sigma}I_{\rho\sigma})(\delta_{\nu\mu} + \varepsilon_{\nu\mu})\beta_\mu(1 - \frac{1}{2}\varepsilon_{\rho\sigma}I_{\rho\sigma}) = \beta_\nu, \quad (49)$$

expanding Eq. (50) to one order of ε , we obtain

$$\frac{1}{2}\varepsilon_{\rho\sigma}(I_{\rho\sigma}\beta_\nu - \beta_\nu I_{\rho\sigma}) = -\varepsilon_{\nu\mu}\beta_\mu. \quad (50)$$

Eq. (50) give the relation between the infinitesimal operator $I_{\rho\sigma}$ and matrix β_ν , i.e., the transformation L is existential. The spinor wave equationS (20) and(24) are covariant.

5. Conclusion

In classical electromagnetism theory, the 4-vector potential A_μ satisfies the differential equation of space-time two order. In this paper, we give the spinor wave equations of free and unfree photon, which are the differential equation of space-time one order, and we prove the spinor wave equations of free photon is covariant. For the unfree photon, the spinor wave equations can be used to studied the interaction between photon and medium, photonic crystals and so on.

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